There are Significantly More Nonnegative Polynomials than Sums of Squares

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Abstract

We investigate the quantitative relationship between nonnegative polynomials and sums of squares of polynomials. We show that if the degree is fixed and the number of variables grows then there are significantly more nonnegative polynomials than sums of squares. More specifically, we take compact bases of the cone of nonnegative polynomials and the cone of sums of squares and derive bounds for the volumes of the bases. If the degree is greater than 2 then we show that the ratio of the volumes of the bases, raised to the power reciprocal to the ambient dimension, tends to 0 as the number of variables tends to infinity.

1 Introduction

Let $P_{n,2k}$ be the vector space of real homogeneous polynomials in n variables of degree $2k$. The study of the relationship between nonnegative polynomials in $P_{n,2k}$ and sums of squares of polynomials of degree k was initiated by Hilbert. He showed that for the cases $n = 2$, $k = 1$ and, $n = 3$ and $k = 2$, a nonnegative polynomial is necessarily a sum of squares; in all other cases there exist nonnegative polynomials that are not sums of squares. Hilbert's proof of existence of nonnegative polynomials that are not sums of squares was non-constructive [6]. The first such explicit polynomials were constructed only fifty years later by Motzkin in the 1940's. There are currently several known families of non-negative polynomials that are not sums of squares [3],[9].

There remains however a natural question, which we call, the *Quantitative Sums of Squares Problem*:

Are there significantly more nonnegative polynomials than sums of squares of polynomials?

The known examples of nonnegative polynomials that are not sums of squares live either on the boundary of the cone of non-negative polynomials or very close to it. The difficulty in constructing such polynomials explicitly leads naturally to asking whether they are pathological examples, while the bulk of nonnegative polynomials are sums of squares.

In this paper we show that if the degree $2k$ is fixed and we let the number of variables grow then there are significantly more nonnegative polynomials than sums of squares. One could hope that sums of squares take up a perhaps small but constant portion of the cone of nonnegative polynomials, but we will show this is not the case. As a corollary it follows that if the number of variables is large then nonnegative polynomials that are not sums of squares are far more "normal" than sums of squares themselves.

The Quantitative Sums of Squares Problem also has significance from the point of view of computational complexity. Using the tools of semidefinite programming one can efficiently compute whether a given polynomial is a sum of squares [7]. However, determining whether a polynomial is nonnegative is NP-hard for $k \geq 2$ [2, Part 1]. Therefore we can ask how much we lose by testing for sums of squares instead of nonnegativity. There has been some experimental evidence that for certain subsets of the cone of nonnegative polynomials sums of squares approximate nonnegative polynomials rather well [7]. However, it follows from the results of this paper that if the number of variables is large compared to the degree then there are far fewer sums of squares than nonnegative polynomials, and thus relaxation of testing for nonnegativity to testing for sums of squares does not work well.

2 Main Theorems

We begin by introducing some notation. Nonnegative polynomials and sums of squares form full-dimensional convex cones in $P_{n,2k}$. Let $C(= C_{n,2k})$ be the cone of nonnegative polynomials,

$$C = \left\{ f \in P_{n,2k} \mid f(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n \right\}.$$

Let $Sq(=Sq_{n,2k})$ be the cone of sums of squares,

$$Sq = \left\{ f \in P_{n,2k} \;\middle|\; f = \sum_i f_i^2 \quad \text{for some} \quad f_i \in P_{n,k} \right\}.$$

We work with the following inner product on $P_{n,2k}$,

$$\langle f\,,g\rangle = \int_{S^{n-1}} fg\,d\sigma,$$

where σ is the rotation invariant probability measure on S^{n-1}.

In order to compare the cones C and Sq we take compact bases. Let $M(=M_{n,2k})$ be the hyperplane of all forms in $P_{n,2k}$ with integral 0 on the unit sphere S^{n-1}:

$$M = \left\{ f \in P_{n,2k} \;\middle|\; \int_{S^{n-1}} f\,d\sigma = 0 \right\}.$$

We use D_M to denote the dimension of M, S_M to denote the unit sphere in M and B_M to denote the unit ball in M.

Let r^{2k} in $P_{n,2k}$ be the following polynomial

$$r^{2k} = (x_1^2 + \ldots + x_n^2)^k.$$

We define compact convex bodies $\widetilde{C}$ and $\widetilde{Sq}$ as the sets of all forms f in M such that $f + r^{2k}$ lies in the respective cone:

$$\widetilde{C} = \{f \in M \mid f + r^{2k} \in C\}, \qquad \widetilde{Sq} = \{f \in M \mid f + r^{2k} \in Sq\},$$

Convex bodies $\widetilde{C}$ and $\widetilde{Sq}$ are sections of their respective cones with the hyperplane of forms of integral one, which are translated into M by subtracting r^{2k}.

The main result of this paper are the two estimates below:

Theorem 2.1. *There is the following lower bound on the volume of $\widetilde{C}$:*

$$\left(\frac{Vol\,\widetilde{C}}{Vol\,B_M}\right)^{\frac{1}{D_M}} \geq \frac{1}{2\sqrt{4k+2}}\,n^{-1/2}.$$

Theorem 2.2. *There is the the following upper bound on the volume of $\widetilde{Sq}$:*

$$\left(\frac{Vol\,\widetilde{Sq}}{Vol\,B_M}\right)^{\frac{1}{D_M}} \leq \frac{4^{2k}(2k)!\sqrt{24}}{k!}\,n^{-k/2}.$$

From the above two theorems we trivially obtain the following corollary which allows us to compare C and Sq:

Corollary 2.3. *There exists a constant $c(k)$ dependent only on k such that*

$$\left(\frac{Vol\,\widetilde{C}}{Vol\,\widetilde{Sq}}\right)^{\frac{1}{D_M}} \geq c(k)n^{(k-1)/2}.$$

It suffices to take

$$c(k) = \frac{k!}{2(2k)!4^{2k}\sqrt{24(4k+2)}}.$$

We note that if $k = 1$ then $\widetilde{C} = \widetilde{Sq}$ and indeed the bound of Corollary 2.3 becomes just $c(1)$. However, if we fix $k \geq 2$ then there are clearly far more nonnegative polynomials than sums of squares.

3 Volumes of Nonnegative Polynomials

In this section we prove Theorem 2.1. For a real vector space V with the unit sphere S_V and a function $f : V \to \mathbb{R}$ we use $||f||_p$ to denote the L^p norm of f:

$$||f||_p = \left(\int_{S_V} |f|^p \, d\mu\right)^{1/p} \qquad \text{and} \qquad ||f||_\infty = \max_{x \in S_V} |f(x)|.$$

We begin by observing that $\widetilde{C}$ is a convex body in $M_{n,2k}$ with origin in its interior and the boundary of $\widetilde{C}$ consists of polynomials with minimum -1 on S^{n-1}. Therefore the gauge G_C of $\widetilde{C}$ is given by:

$$G_C(f) = |\min_{v \in S^{n-1}} f(v)\,|.$$

By using integration in polar coordinates in M it is easy to obtain the following expression for the volume of $\widetilde{C}$,

$$\left(\frac{\text{Vol}\,\widetilde{C}}{\text{Vol}\;B_M}\right)^{\frac{1}{D_M}} = \left(\int_{S_M} G_C^{-D_M} \, d\mu\right)^{\frac{1}{D_M}}, \tag{3.0.1}$$

where μ is the rotation invariant probability measure on S_M. The relationship (3.0.1) holds for any convex body with origin in its interior [8, p. 91].

We interpret the right hand side of (3.0.1) as $||G_C^{-1}||_{D_M}$, and by Hölder's inequality

$$||G_C^{-1}||_{D_M} \geq ||G_C^{-1}||_1.$$

Thus,

$$\left(\frac{\operatorname{Vol} \widetilde{C}}{\operatorname{Vol} B_M}\right)^{\frac{1}{D_M}} \geq \int_{S_M} G_C^{-1}\, d\mu.$$

By applying Jensen's inequality [5, p.150], with convex function $y = 1/x$ it follows that,

$$\int_{S_M} G_C^{-1}\, d\mu \geq \left(\int_{S_M} G_C\, d\mu\right)^{-1}.$$

Hence we see that

$$\left(\frac{\operatorname{Vol} \widetilde{C}}{\operatorname{Vol} B_M}\right)^{\frac{1}{D_M}} \geq \left(\int_{S_M} |\min f|\, d\mu\right)^{-1}.$$

Clearly, for all $f \in P_{n,2k}$

$$||f||_\infty \geq |\min f|.$$

Therefore,

$$\left(\frac{\operatorname{Vol} \widetilde{C}}{\operatorname{Vol} B_M}\right)^{\frac{1}{D_M}} \geq \left(\int_{S_M} ||f||_\infty\, d\mu\right)^{-1}.$$

The proof of Theorem 2.1 is now completed by the following estimate.

Theorem 3.1. *Let S_M be the unit sphere in M and let μ be the rotation invariant probability measure on S_M. Then the following inequality for the average L^∞ norm over S_M holds:*

$$\int_{S_M} ||f||_\infty\, d\mu \leq 2\sqrt{2n(2k+1)}.$$

Proof. It was shown by Barvinok in [1] that for all $f \in P_{n,2k}$,

$$||f||_\infty \leq \binom{2kn+n-1}{2kn}^{\frac{1}{2n}} ||f||_{2n}.$$

By applying Stirling's formula we can easily obtain the bound

$$\binom{2kn+n-1}{2kn}^{\frac{1}{2n}} \leq 2\sqrt{2k+1}.$$

Therefore it suffices to estimate the average L^{2n} norm, which we denote by A:

$$A = \int_{S_M} ||f||_{2n} \, d\mu.$$

Applying Hölder's inequality we observe that

$$A = \int_{S_M} \left(\int_{S^{n-1}} f^{2n}(x) \, d\sigma \right)^{\frac{1}{2n}} d\mu \leq \left(\int_{S_M} \int_{S^{n-1}} f^{2n}(x) \, d\sigma \, d\mu \right)^{\frac{1}{2n}}.$$

By interchanging the order of integration we obtain

$$A \leq \left(\int_{S^{n-1}} \int_{S_M} f^{2n}(x) \, d\mu \, d\sigma \right)^{\frac{1}{2n}}. \tag{3.1.1}$$

We now note that by symmetry of M

$$\int_{S_M} f^{2n}(x) \, d\mu,$$

is the same for all $x \in S^{n-1}$. Therefore we see that in (3.1.1) the outer integral is redundant and thus

$$A \leq \left(\int_{S_M} f^{2n}(v) \, d\mu \right)^{\frac{1}{2n}}, \qquad \text{where } v \text{ is any vector in } S^{n-1}. \tag{3.1.2}$$

For $v \in \mathbb{R}^n$ the functional

$$\lambda_v : M \longrightarrow \mathbb{R}, \qquad \lambda_v(f) = f(v),$$

is linear and therefore there exists a form $q_v \in M$ such that

$$\lambda_v(f) = \langle q_v \, , f \rangle.$$

Rewriting (3.1.2) we see that

$$A \leq \left(\int_{S_M} \langle f \, , q_v \rangle^{2n} \, d\mu \right)^{\frac{1}{2n}}. \tag{3.1.3}$$

There are explicit descriptions of the polynomials q_v, see for example [11], we will only need the property that for $v \in S^{n-1}$

$$|| \, q_v||_2 = \sqrt{D_M}.$$

This can also be deduced by abstract representation theoretic considerations.

We observe that

$$\int_{S_M} \langle f\,, q_v \rangle^{2n}\, d\mu = (D_M)^n\ \frac{\Gamma(n+\frac{1}{2}\,)\,\Gamma(\frac{1}{2}D_M)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}D_M+n)}.$$

We substitute this into (3.1.3) to obtain,

$$A \le \left((D_M)^n\ \frac{\Gamma(n+\frac{1}{2}\,)\,\Gamma(\frac{1}{2}D_M)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}D_M+n)} \right)^{\frac{1}{2n}}.$$

Since

$$\left(\frac{\Gamma(\frac{1}{2}D_M)}{\Gamma(\,\frac{1}{2}D_M+n)} \right)^{\frac{1}{2n}} \le \sqrt{\frac{2}{D_M}} \qquad \text{and} \qquad \left(\frac{\Gamma(n+1/2\,)}{\sqrt{\pi}} \right)^{\frac{1}{2n}} \le n^{1/2},$$

we see that

$$A \le (2n)^{1/2}.$$

The theorem now follows. $\square$

4 Volumes of Sums of Squares

In this section we prove Theorem 2.2. Let us begin by considering the support function of $\widetilde{Sq}$, which we call $L_{\widetilde{Sq}}$:

$$L_{\widetilde{Sq}}(f) = \max_{g\,\in\,\widetilde{Sq}} \langle f\,, g\rangle.$$

The average width $W_{\widetilde{Sq}}$ of $\widetilde{Sq}$ is given by

$$W_{\widetilde{Sq}} = 2\int_{S_M} L_{\widetilde{Sq}}\, d\mu.$$

We now recall Urysohn's Inequality [10, p.318] which applied to $\widetilde{Sq}$ gives

$$\left(\frac{\mathrm{Vol}\,\widetilde{Sq}}{\mathrm{Vol}\,B_M} \right)^{\frac{1}{D_M}} \le \frac{W_{\widetilde{Sq}}}{2}. \qquad (4.0.1)$$

Therefore it suffices to obtain an upper bound for $W_{\widetilde{Sq}}$.

Let $S_{P_{n,k}}$ denote the unit sphere in $P_{n,k}$. We observe that extreme points of $\widetilde{Sq}$ have the form

$$g^2 - r^{2k} \qquad \text{where} \qquad g \in P_{n,k} \qquad \text{and} \qquad \int_{S^{n-1}} g^2 \, d\sigma = 1.$$

For $f \in M$,

$$\langle f \,, r^{2k} \rangle = \int_{S^{n-1}} f \, d\sigma = 0,$$

and therefore,

$$L_{\widetilde{Sq}}(f) = \max_{g \in S_{P_{n,k}}} \langle f \,, g^2 \rangle.$$

We now introduce a norm on $P_{n,2k}$, which we denote $|| \; ||_{sq}$:

$$||f||_{sq} = \max_{g \in S_{P_{n,k}}} |\langle f \,, g^2 \rangle|.$$

It is clear that

$$L_{Sq}(f) \leq ||f||_{Sq}.$$

Therefore by (4.0.1) it follows that

$$\left(\frac{\operatorname{Vol} \widetilde{Sq}}{\operatorname{Vol} B_M} \right)^{\frac{1}{D_M}} \leq \int_{S_M} ||f||_{sq} \, d\mu.$$

The proof of Theorem 2.2 is reduced to the estimate below.

Theorem 4.1. *There is the following bound for the average $|| \; ||_{sq}$ over S_M:*

$$\int_{S_M} ||f||_{sq} \, d\mu \leq \frac{4^{2k} (2k)! \sqrt{24}}{k!} \, n^{-k/2}.$$

Proof. For $f \in P_{n,2k}$ we introduce a quadratic form H_f on $P_{n,k}$:

$$H_f(g) = \langle f \,, g^2 \rangle \qquad \text{for} \qquad g \in P_{n,k}.$$

We note that

$$||f||_{sq} = \max_{g \in S_{P_{n,k}}} |\langle f \,, g \rangle| = ||H_f||_{\infty}.$$

We bound $||H_f||_\infty$ by a high L^{2p} norm of H_f. Since H_f is a form of degree 2 on the vector space $P_{n,k}$ of dimension $D_{n,k}$ it follows by the inequality of Barvinok in [1] applied in the same way as in the proof of Theorem 2.1 that

$$||H_f||_\infty \le 2\sqrt{3}\,||H_f||_{2D_{n,k}}.$$

Therefore it suffices to estimate:

$$A = \int_{S_M} ||H_f||_{2D_{n,k}}\, d\mu = \int_{S_M} \left(\int_{S_{P_{n,k}}} \langle f\,, g^2 \rangle^{\,2D_{n,k}}\, d\sigma(g)\, d\mu(f) \right)^{\frac{1}{2D_{n,k}}}.$$

We apply Hölder's inequality to see that

$$A \le \left(\int_{S_M} \int_{S_{P_{n,k}}} \langle f\,, g^2 \rangle^{\,2D_{n,k}}\, d\sigma(g)\, d\mu(f) \right)^{\frac{1}{2D_{n,k}}}.$$

By interchanging the order of integration we obtain

$$A \le \left(\int_{S_{P_{n,k}}} \int_{S_M} \langle f\,, g^2 \rangle^{\,2D_{n,k}}\, d\mu(f)\, d\sigma(g) \right)^{\frac{1}{2D_{n,k}}}. \tag{4.1.1}$$

Now we observe that the inner integral

$$\int_{S_M} \langle f\,, g^2 \rangle^{\,2D_{n,k}}\, d\mu(f),$$

clearly depends only on the length of the projection of g^2 into M. Therefore we have

$$\int_{S_M} \langle f\,, g^2 \rangle^{\,2D_{n,k}}\, d\mu(f) \le\; ||g^2||_2^{2D_{n,k}} \int_{S_M} \langle f\,, p \rangle^{2D_{n,k}}\, d\mu(f) \quad \text{for any} \quad p \in S_M.$$

We observe that

$$||g^2||_2 = (||g||_4)^2 \qquad \text{and} \qquad ||g||_2 = 1.$$

By a result of Duoandikoetxea [4] Corollary 3 it follows that

$$||g^2||_2 \le 4^{2k}.$$

Hence we obtain

$$\int_{S_M} \langle f\,, g^2\rangle^{\,2D_{n,k}}\, d\mu(f) \le 4^{4kD_{n,k}} \int_{S_V} \langle f\,, p\rangle^{2D_{n,k}}\, d\mu(f).$$

We note that this bound is independent of g and substituting into (4.1.1) we get

$$A \le 4^{2k}\left(\int_{S_V} \langle f\,, p\rangle^{2D_{n,k}}\, d\mu(f)\right)^{\frac{1}{2D_{n,k}}}.$$

Since $p \in S_M$ we have

$$\int_{S_M} \langle f\,, p\rangle^{2D_{n,k}}\, d\mu(f) = \frac{\Gamma(D_{n,k}+\frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\pi}\,\Gamma(D_{n,k}+\frac{1}{2}D_M)}.$$

We use the following easy inequalities:

$$\left(\frac{\Gamma(\frac{1}{2}D_M)}{\Gamma(D_{n,k}+\frac{1}{2}D_M)}\right)^{\frac{1}{2D_{n,k}}} \le \sqrt{\frac{2}{D_M}}$$

and

$$\left(\frac{\Gamma(D_{n,k}+\frac{1}{2})}{\sqrt{\pi}}\right)^{\frac{1}{2D_{n,k}}} \le \sqrt{D_{n,k}},$$

to see that

$$A \le 4^{2k}\sqrt{\frac{2D_{n,k}}{D_M}}.$$

We now recall that

$$D_{n,k} = \binom{n+k-1}{k} \qquad \text{and} \qquad D_M = \binom{n+2k-1}{2k} - 1.$$

Therefore

$$\sqrt{\frac{D_{n,k}}{D_M}} \le \frac{(2k)!}{k!}\, n^{-k/2}.$$

Thus

$$A \le \frac{4^k (2k)!\sqrt{2}}{k!}\, n^{-k/2}.$$

The theorem now follows. $\square$

5 Remarks

The estimates of Theorems 2.1 and 2.2 for the volumes of nonnegative polynomials and sums of squares are asymptotically exact if the degree is fixed. The proof, however, is more technical and shall be discussed elsewhere.

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